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Contraexamples in Difference Posets and Orthoalgebras

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We show that every orthoalgebra (difference orthoposet) uniquely determines a difference orthoalgebraic structure. We give examples of posets on which there exist more than one difference operation. In spite of that, every finite chain is a uniquely determined difference poset. On a difference poset there need not exist any orthoalgebraic operation, but the category of difference orthoposets is isomorphic with the category of orthoalgebras. But a difference poset which is also an orthoposet need not be a difference orthoposet. Moreover, there exist complete lattices on which there does not exist any difference operation. Finally, we show that difference operations and orthoalgebraic operations need not be extendable on a MacNeille completion of the base poset.

1. PRELIMINARIES

Let us start with the basic notions of our considerations.

Let a binary relation $\leq$ on a nonvoid set P be a partial ordering; then a pair $(P, \leq)$ is called a *poset*. A *chain* is a poset in which every two elements a, b are *comparable*, i.e. $a \leq b$ or $b \leq a$.

A structure $(P, \leq, ^{\perp}, 0, 1)$ is called an *orthoposet* if $(P, \leq)$ is a poset and the unary operation $\perp: a \in P \to a^{\perp} \in P$ is such that for every $x, y \in P$:

(opi) $(x^{\perp})^{\perp} = x$.
(opii) $x \leq y$ implies $y^{\perp} \leq x^{\perp}$.
(opiii) $x \vee x^{\perp} = 1$.
(opiv) $0^{\perp} = 1$.

An orthoposet P is called an *orthomodular poset* if $x \vee y$ exists for any pair $x, y \in P$ such that $x \leq y^{\perp}$, and the orthomodular law is valid in P, i.e.,

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$y = x \vee (x^{\perp} \wedge y)$ for every $x, y \in P$ such that $x \leq y$. An orthomodular poset which is a lattice is called an *orthomodular lattice*.

We say that elements x, y of an orthoposet P are *orthogonal* if $x \leq y^{\perp}$. The set $M \subseteq P$ is said to be orthogonal if every two different elements of M are orthogonal.

In what follows, for a partially defined operation "$\bigcirc$" on a nonempty set P and any $a, b, c \in P$ we write $a \bigcirc b = c$ if $a \bigcirc b$ is defined and $c = a \bigcirc b$.

A structure $(P, \oplus, 0, 1)$ is called an *orthoalgebra* if $0, 1$ are two distinguished elements and $\oplus$ is a partially defined binary operation on P which satisfies the following conditions for any $a, b, c \in P$:

(oai) $b \oplus a = a \oplus b$ if $a \oplus b$ is defined.
(oaii) $(a \oplus b) \oplus c = a \oplus (b \oplus c)$ if $b \oplus c$ and $a \oplus (b \oplus c)$ are defined.
(oaiii) For every $a \in P$ there exists a unique $b \in P$ such that $a \oplus b = 1$.
(oaiv) If $a \oplus a$ is defined, then $a = 0$.

Orthoalgebras are algebraic systems that generalize Boolean algebras and orthomodular posets [see Randall and Foulis (1981) and Foulis and Pták (n.d.) and references therein].

Further generalizations for the abstract model of the quantum logic approach to the foundation of quantum mechanics are difference posets (Kôpka and Chovanec, n.d.; Navara and Pták, n.d.; Foulis and Pták, n.d.).

A structure $(P, \leq, \ominus, 0, 1)$ is called a *difference poset* (abbreviated D-poset or DP) if $(P, \leq)$ is a poset with a least element 0 and a greatest element 1 and with a partially defined binary operation $\ominus$ such that for any $a, b, c \in P$ the following are satisfied:

(dpi) $b \ominus a$ is defined iff $a \leq b$.
(dpii) $a \ominus 0 = a$.
(dpiii) If $a \leq b \leq c$, then $c \ominus b \leq c \ominus a$ and $(c \ominus a) \ominus (c \ominus b) = b \ominus a$.

Typical examples of difference posets are orthomodular lattices and orthoalgebras [see Navara and Pták (n.d.) and Kôpka and Chovanec (n.d.) for further examples].

A structure $(P, \leq, ^{\perp}, \ominus, 0, 1)$ is called a *difference orthoposet* (abbreviated D-orthoposet or DOP) if the following conditions are satisfied:

(dopi) $(P, \leq, \ominus, 0, 1)$ is a D-poset.
(dopii) $(P, \leq, ^{\perp}, 0, 1)$ is an orthoposet.
(dopiii) $a^{\perp} = 1 \ominus a$ for every $a \in P$.

2. EXAMPLES

The easy proof of the following lemma is left to the reader.

Lemma 2.1. Let $(P, \leq, \ominus, 0, 1)$ be a D-poset. Then the following conditions are satisfied for any $a, b, c \in P$:

(i) $b \ominus a \leq b$ and $b \ominus (b \ominus a) = a$ for every $a \leq b$.
(ii) $a \notin \{0, 1\}$ implies $1 \ominus a \notin \{0, 1\}$.
(iii) $0 < a < b < 1$ implies $0 < 1 \ominus b < 1 \ominus a < 1$ and $0 < b \ominus a < b$.
(iv) $b \ominus a = 0$ if and only if $a = b$.
(v) $b \ominus a = b$ implies $a = 0$.
(vi) $c \ominus a = c \ominus b$ implies $a = b$.
(viii) $a \ominus c = b \ominus c$ implies $a = b$.
(viii) $0 < a < b < c < 1$ implies $0 < c \ominus b < c \ominus a < c < 1$.
(ix) $a \leq b \leq c$ implies $b \ominus a \leq c \ominus a$ and $(c \ominus a) \ominus (b \ominus a) = c \ominus b$.
(x) $b \leq c \ominus a$ implies $(c \ominus a) \ominus b = (c \ominus b) \ominus a$.
(xi) $a \leq c \ominus b$ implies $b \leq c \ominus a$.

Corollary 2.2. Every finite chain is a uniquely defined D-poset.

Proof. Let a poset P be a finite chain: $0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = 1$. Then, in view of Lemma 1.1, for every $i, k \in \{1, 2, \ldots, n\}$ and any difference operation $\ominus$ on P we have:

$$x_k \ominus 0 = x_k$$

$$x_k \ominus x_k = 0$$

$$x_k \ominus x_i \text{ is not defined for every } i > k$$

$$x_k \ominus x_i = x_{k-i} \text{ for every } i < k$$

The last is obvious from the fact that $0 < x_1 < x_2 < \cdots < x_{k-1} < x_k < 1$ implies that

$$0 = x_k \ominus x_k < x_k \ominus x_{k-1} < \cdots < x_k \ominus x_2 < x_k \ominus x_1 < x_k \ominus 0 = x_k$$

In what follows we shall consider three examples of posets (Figs. 1–3).

Example 2.3. Figure 1 shows an orthoposet P which does not become a D-orthoposet by putting $1 \ominus x = x^\perp$ for every $x \in P$.

Suppose the contrary. Then in view of Lemma 1.1 we have $b^\perp \ominus a \leq b^\perp$ and $b^\perp \ominus a \notin \{0, 1, b^\perp\}$, hence $b^\perp \ominus a = a$. Similarly $a^\perp \ominus b = b$.

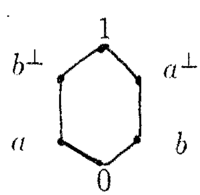

Fig. 1

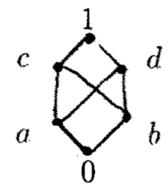

Fig. 2

Furthermore, by (dpiii) $b \leq a^{\perp} \leq 1$ implies $(1 \ominus b) \ominus (1 \ominus a^{\perp}) = a^{\perp} \ominus b$. It follows that $b^{\perp} \ominus a = a^{\perp} \ominus b$ and hence $a = b$; this is a contradiction.

In spite of this fact, if we put $1 \ominus a = b^{\perp}$, $1 \ominus b = a^{\perp}$, $1 \ominus a^{\perp} = b$, and $1 \ominus b^{\perp} = a$, then the poset P in Fig. 1 becomes a D-poset in which $a^{\perp} \ominus b = b$ and $b^{\perp} \ominus a = a$. Moreover $x \ominus 0 = x$ and $x \ominus x = 0$ for every $x \in P$. *Hence this structure* $(P, \leq, {}^{\perp}, \ominus, 0, 1)$ *satisfies the conditions (dopi) and (dopii) but not (dopiii) of D-orthoposets.*

Example 2.4. For the poset P in Fig. 3 there does not exist any difference operation $\ominus$ with the properties (dpi)–(dpiii) of D-posets.

Suppose the contrary. Then in view of Lemma 2.1 we have

$$0 < b < e < c < 1 \quad \text{implies} \quad 0 < c \ominus e < c \ominus b < c < 1$$

and

$$0 < a < e < c < 1 \quad \text{implies} \quad 0 < c \ominus e < c \ominus a < c < 1$$

It follows that $c \ominus a = c \ominus b = e$. Thus by property (x) of Lemma 2.1 we have

$$0 = (c \ominus a) \ominus e = (c \ominus e) \ominus a$$

$$0 = (c \ominus b) \ominus e = (c \ominus e) \ominus b$$

Hence $c \ominus e = a = b$, which is a contradiction.

Example 2.5. For the poset P in Fig. 2 there exist four different operations $\ominus_i$ such that $(P, \leq, \ominus_i, 0, 1)$, $i \in \{1, 2, 3, 4\}$, is a D-poset.

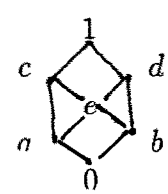

Fig. 3

Proof. Evidently if a difference operation $\ominus$ on a poset P exists, then we have

$$x \ominus x = 0, \qquad x \ominus 0 = x \qquad \text{for every } x \in P$$

and $x \ominus y$ is not defined for every $x, y \in P$ with $x < y$. Thus, it remains to choose

$$c \ominus a, c \ominus b \in \{a, b\} \text{ with } c \ominus a \neq c \ominus b$$

$$d \ominus a, d \ominus b \in \{a, b\} \text{ with } d \ominus a \neq d \ominus b$$

$$1 \ominus a, 1 \ominus b \in \{c, d\} \text{ with } 1 \ominus a \neq 1 \ominus b$$

$$1 \ominus c, 1 \ominus d \in \{a, b\} \text{ with } 1 \ominus c \neq 1 \ominus d$$

and such that [in view of (xi), Lemma 2.1]

$$1 \ominus a = c \text{ implies } 1 \ominus c = a$$

$$1 \ominus a = d \text{ implies } 1 \ominus d = b$$

It is clear that we have four different possibilities for operation $\ominus$. In all these possibilities the following conditions are satisfied:

$$0 < b < c < 1 \text{ implies } 0 < 1 \ominus c < 1 \ominus b < 1 \text{ and } (1 \ominus b) \ominus (1 \ominus c) = c \ominus b$$

$$0 < a < d < 1 \text{ implies } 0 < 1 \ominus d < 1 \ominus a < 1 \text{ and } (1 \ominus a) \ominus (1 \ominus d) = d \ominus a$$

$$0 < a < c < 1 \text{ implies } 0 < 1 \ominus c < 1 \ominus a < 1 \text{ and } (1 \ominus a) \ominus (1 \ominus c) = c \ominus a$$

$$0 < b < d < 1 \text{ implies } 0 < 1 \ominus d < 1 \ominus b < 1 \text{ and } (1 \ominus b) \ominus (1 \ominus d) = d \ominus b$$

3. ORTHOALGEBRAS AND D-ORTHOPOSETS AS D-ORTHOALGEBRAS

Connections between orthoalgebras and orthomodular posets (lattices) have been studied by Navara and Pták (n.d.). Using their results, we are going to show that the category of orthoalgebras is isomorphic to the category of D-orthoposets and each of them uniquely determines a D-orthoalgebraic structure on the base set. From these facts we obtain necessary and sufficient conditions for difference operations $\ominus$ and orthoalgebraic operations $\oplus$ to be extendable to the MacNeille completion of the base poset.

Lemma 3.1. Let $(P, \leq, \ominus, 0, 1)$ be a D-poset. The following conditions are equivalent:

(i) For every $b \in P$: $1 \ominus b \leq b$ implies $b = 1$.
(ii) For every $a \in P$: $a \leq 1 \ominus a$ implies $a = 0$.
(iii) $(P, \leq, ^{\perp}, \ominus, 0, 1)$ with orthocomplementation $^{\perp}: P \to P$ defined by $a^{\perp} = 1 \ominus a$ is a D-orthoposet.

Proof. (i) $\Leftrightarrow$ (ii): In view of (dpiii), for every $a \in P$ we have $a \leq 1 \ominus a$ iff $1 \ominus (1 \ominus a) \leq 1 \ominus a$, and we use the fact that $1 \ominus a = 1$ iff $a = 0$ by Lemma 2.1.

(i) $\Rightarrow$ (iii): For any $a \in P$ we have $(a^{\perp})^{\perp} = 1 \ominus (1 \ominus a) = a$. If $a, b \in P$ with $a \leq b$, then by (dpiii) we have $1 \ominus b \leq 1 \ominus a$, hence $b^{\perp} \leq a^{\perp}$. Moreover, for any $a \in P$ and every $b \in P$ such that $a \leq b$ and $a^{\perp} \leq b$ we have $1 \ominus b \leq 1 \ominus a = a^{\perp} \leq b$ and thus the supremum $a \vee a^{\perp} = 1$. Finally, $0^{\perp} = 1 \ominus 0 = 1$.

(iii) $\Rightarrow$ (i): This is obvious.

A D-poset $(P, \leq, \ominus, 0, 1)$ with $a \leq 1 \ominus a \Rightarrow a = 0$ for every $a \in P$ is called a regular D-poset (Navara and Pták, n.d.).

Definition 3.2. A structure $(P, \leq, {}^{\perp}, \ominus, \oplus, 0, 1)$ is called a *difference orthoalgebra* (abbreviated D-orthoalgebra, or DOA) if the following conditions are satisfied:

(doai) $(P, \leq, {}^{\perp}, \ominus, 0, 1)$ is a D-orthoposet.
(doaii) $(P, \oplus, 0, 1)$ is an orthoalgebra.
(doaiii) For every $a, b \in P$ with $a \leq 1 \ominus b$: $a \oplus b = 1 \ominus [(1 \ominus a) \ominus b]$.

In this case we say that the orthoalgebra $(P, \oplus, 0, 1)$ and D-orthoposet $(P, \leq, {}^{\perp}, \ominus, 0, 1)$ *belong to one another.*

Proposition 3.3. Let P be a set and 0, 1 be two distinguished elements of P. Then every D-orthoposet structure on P uniquely determines an orthoalgebraic structure on P, and every orthoalgebraic structure on P uniquely determines a D-orthoposet structure on P, such that they belong to one another; i.e., they both uniquely determine the same D-orthoalgebraic structure.

Proof. (1) Let $(P, \leq, {}^{\perp}, \ominus, 0, 1)$ be a D-orthoposet. For every $a, b \in P$ with $a \leq 1 \ominus b$ we define $a \oplus b = 1 \ominus [(1 \ominus a) \ominus b]$. Then in view of Lemma 3.1, $(P, \oplus, 0, 1)$ is an orthoalgebra (Navara and Pták, n.d., Theorem 1.11).

(2) Let $(P, \oplus, 0, 1)$ be an orthoalgebra. We define for any $a, b \in P$: $a \leq b$ iff there exists $c \in P$ with $b = a \oplus c$ and then we put $b \ominus a = c$.

Moreover, we put $a^{\perp} = 1 \ominus a$ for every $a \in P$.

Then $(P, \leq, {}^{\perp}, \ominus, 0, 1)$ is a D-poset (Navara and Pták, n.d., Proposition 1.9) and in view of Lemma 3.1 it is a D-orthoposet. Further, let $a, b \in P$ with $a \leq 1 \ominus b$. Then

$$1 \ominus b = a \oplus [(1 \ominus b) \ominus a] \qquad \text{by the definition of operation } \ominus$$

It follows that

$$1 = b \oplus (1 \ominus b) = b \oplus \{a \oplus [(1 \ominus b) \ominus a]\} = (b \oplus a) \oplus [(1 \ominus b) \ominus a]$$

Hence $(1 \ominus b) \ominus a = (b \oplus a)^{\perp}$ and thus $b \oplus a = 1 \ominus [(1 \ominus b) \ominus a]$.

Proposition 3.4. Let $(P, \leq, {}^{\perp}, \ominus, \oplus, 0, 1)$ be a D-orthoalgebra. Then the following conditions are equivalent:

(i) $(P, \leq, {}^{\perp}, 0, 1)$ is an orthomodular poset.
(ii) The supremum $a \vee b$ exists in P for every $a, b \in P$ with $a \leq b^{\perp}$.

Proof. (i) $\Rightarrow$ (ii): This is obvious.

(ii) $\Rightarrow$ (i): Suppose that $a, b \in P, a \leq b$. Then $a \leq (b^{\perp})^{\perp}$ implies that $a \vee b^{\perp}$ exists in P and hence the infimum $a^{\perp} \wedge b$ exists in P. Moreover, we have $b \ominus a \leq b$ and

$$a \leq b \leq 1 \Rightarrow (1 \ominus a) \ominus (1 \ominus b) = b \ominus a \Rightarrow b \ominus a \leq 1 \ominus a = a^{\perp}$$

We obtain $b \ominus a \leq a^{\perp} \wedge b$. It follows that $a^{\perp} \wedge b = 0$ implies $b \ominus a = 0$ and hence $b = a$. We obtain that $(P, \leq, {}^{\perp}, 0, 1)$ is orthomodular (Kalmbach, 1983, p. 27).

Every orthomodular poset $(P, \leq, {}^{\perp}, 0, 1)$ becomes a D-orthoposet putting $b \ominus a = a^{\perp} \wedge b$ for every $a, b \in P$ with $a \leq b$. The following corollary of Proposition 3.3 follows:

Corollary 3.5. Every finitely orthocomplete D-orthoalgebra (i.e., in which the supremum of every two orthogonal elements exists) is uniquely determined by an orthomodular poset.

Example 3.6. The poset in Fig. 1 does not become an orthoalgebra (D-orthoalgebra), because there is no difference operation $\ominus$ on P with property $1 \ominus x = x^{\perp}$.

Example 3.7. The poset P in Fig. 2 does not become an orthoalgebra (D-orthoalgebra), because it is not an orthoposet.

Corollary 3.8. A MacNeille completion MC(P) of a D-orthoposet (D-orthoalgebra) P is again a D-orthoposet (D-orthoalgebra), operations on which extend those of P, if and only if MC(P) is an orthomodular lattice.

Remark 3.9. For a necessary and sufficient condition for orthoposet $(P, \leq, {}^{\perp}, 0, 1)$ to have orthomodular MacNeille completion see Riečanová (n.d.).

4. COMPLETIONS OF D-POSETS AND D-ORTHOALGEBRAS

It is well known that any partially ordered set P can be embedded into its MacNeille completion $\bar{P}$ (or completion by cuts). It has been shown (Schmidt, 1956) that any complete lattice $\bar{P}$ into which P can be supremum-densely and infimum-densely embedded (i.e., every element of $\bar{P}$

is the supremum of elements of the image of P and the infimum of elements of the image of P) is isomorphic to the MacNeille completion of P. For an orthoposet P the MacNeille completion is always a complete ortholattice (Kalmbach, 1983, pp. 255–256) in which orthocomplementation extends that of P.

In this part we show that the MacNeille completion of a D-poset P (D-orthoalgebra P) need not be again a D-poset (D-orthoalgebra) on which the operations extend that of P.

Example 4.1 (Suggested by J. Harding). As we have showed in Example 2.5 for the poset P of Fig. 2, there exist four difference operations. The MacNeille completion of that poset P is a lattice in Fig. 3. But on this lattice there is not any difference operation, as we showed in Example 2.4. Hence: *The MacNeille completion of any D-poset* $(P, \leq, \ominus_i, 0, 1)$, $i = 1, 2, 3, 4$, *from Example 2.5 is not again a D-poset.*

Example 4.2. Consider the orthocomplete orthomodular poset $(P, \leq, ^{\perp}, 0, 1)$ by Fig. 4, where to be understood we identify both atoms a (and hence both coatoms $a^{\perp}$).

Denote by MC(P) the MacNeille completion of that orthomodular poset P. Let $(P, \leq, ^{\perp}, \ominus, \oplus, 0, 1)$ be a D-orthoalgebraic structure on P uniquely determined by the orthomodular poset in Fig. 4. Since $MC(P)$ is a complete ortholattice which is not orthomodular (Kalmbach, 1983, p. 259; Riečanová, n.d.), neither the operation $\ominus$ nor the operation $\oplus$ can be extended to MC(P) (see Proposition 3.4). As M. Navara has proved (unpublished), this orthomodular poset cannot be embedded into any complete orthomodular lattice. Thus, in view of Corollary 3.5, that D-orthoalgebra $(P, \leq, ^{\perp}, \ominus, \oplus, 0, 1)$ cannot be embedded into any complete D-orthoalgebra.

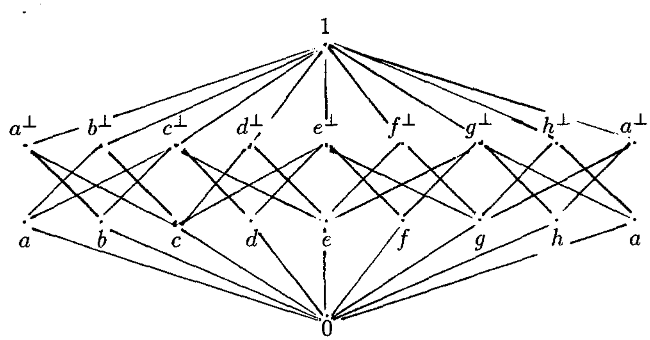

Fig. 4

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